Lesson 3-2

The Graph-Translation Theorem

Vocabulary

transformation
preimage
image
translation

BIG IDEA When the graph of a function is translated vertically or horizontally, its equation changes in a related way.

The Translation Image of a Graph

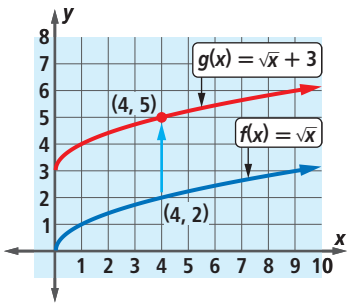
A **transformation** is a one-to-one correspondence between sets of points. Another term for "correspondence" is "mapping." We say that one set, the **preimage**, is mapped onto the other set, the **image**. One type of transformation is a **translation**, which shifts the graph horizontally or vertically. You can quickly describe and interpret translation images of functions of common parent functions. Below, the graph of the parent function f is the preimage. The graph of g is its image under a *vertical translation*.

Mental Math

Consider the data set {2, 2, 3, 4, 6}.

a. Find the mean, median, and mode of the data set.

b. Which measure is least descriptive of the center of the data set?



x	f(x)	g(x)
1	1	4
2	$\sqrt{2}$	$\sqrt{2}+3$
3	$\sqrt{3}$	$\sqrt{3}+3$
4	2	5
5	$\sqrt{5}$	$\sqrt{5}+3$

This translation can also be written as $T(x, y) = (x, y + 3)$ or $T\colon (x, y) \rightarrow (x, y + 3)$, which is read "$(x, y)$ is mapped onto $(x, y + 3)$." The graph of $g(x) = \sqrt{x} + 3$ is the translation image of the graph $f(x) = \sqrt{x}$.

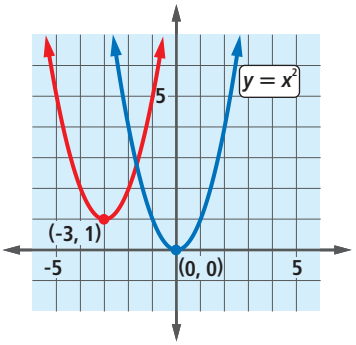
The general translation of the plane translates figures horizontally and vertically at the same time.

Definition of Translation

A **translation** in the plane is a transformation that maps each point (x, y) onto $(x + h, y + k)$, where h and k are constant.

Example 1

The graph of $y = x^2$ is shown at the right, together with its image under a translation T. The point (0, 0), which is the vertex of the preimage, maps onto the vertex (–3, 1) of the image.



a. Find a rule for the translation T.

b. Find the image of (2, 4) under this translation.

Solution

a. The second graph has been obtained from the graph of $y = x^2$ by a translation 3 units to the left and 1 unit up. Thus,

$T(x, y) = (x - 3, y + 1)$.

b. $T(2, 4) = (2 - 3, 4 + 1) = (-1, 5)$

QY1

What is the range of the image function in Example 1?

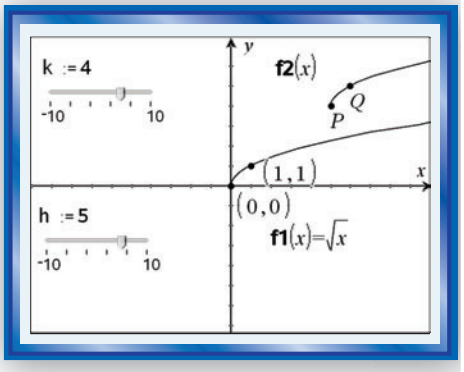
STOP QY1

You can use technology to explore translations.

Activity

MATERIALS Graph variation application supplied by your teacher or the Internet

Consider $f1(x) = \sqrt{x}$. The graph of $f2$ is the image of the graph of $f1$ under a translation of h units horizontally and k units vertically. Points (0, 0) and (1, 1) are on the preimage graph, and P and Q are their images.



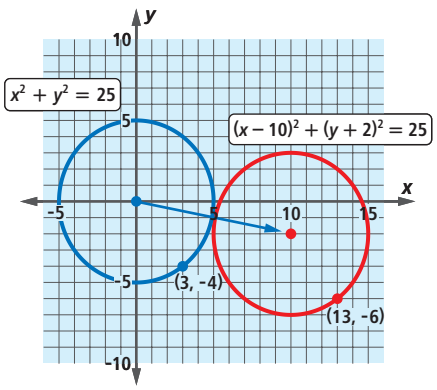
Step 1 The screen at the right shows the instance in which h is 5 and k is 4. Give the coordinates of P and Q.

Step 2 Adjust the sliders so that $h = -3$ and $k = 2$. Give the coordinates of P and Q on the image.

Step 3 Find values of h and k so that the endpoint of the image is at (8, –6).

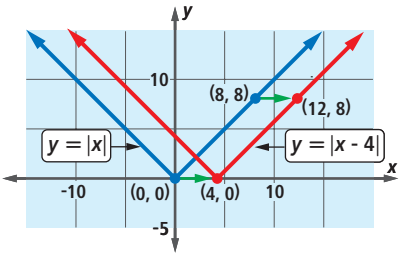
The Graph-Translation Theorem

Graphed in blue at the right is the circle with equation $x^2 + y^2 = 25$ and one point that lies on it, (3, –4). When they are translated 10 units right and 2 units down, the images are the circle with equation $(x - 10)^2 + (y + 2)^2 = 25$ and the point (13, –6), which are shown in red. The preimage point (3, –4) and its image (13, –6) satisfy the equations of their respective circles.



If we let T stand for this translation, then T: $(x, y) \rightarrow (x + 10, y - 2)$. Name the image point (x', y'). Then $x' = x + 10$ and $y' = y - 2$. Solving for x and y gives $x' - 10 = x$ and $y' + 2 = y$. Substitute in the equation $x^2 + y^2 = 25$ and you get $(x' - 10)^2 + (y' + 2)^2 = 25$. It is customary to write the image circle equation without the primes.

There is a direct relationship between replacing a variable expression in an equation and finding the image of a graph under a transformation. Consider the graphs of $f(x) = |x|$ and $g(x) = |x - 4|$ at the right. As the arrow from the point (8, 8) to (12, 8) indicates, the graph of $g(x) = |x - 4|$ is the image of the graph of $f(x) = |x|$ under the translation 4 units to the right, or $(x, y) \rightarrow (x + 4, y)$. Note that adding 4 to each x-coordinate corresponds to replacing x by $x - 4$ in the equation of the preimage. This leads to an important generalization.



Graph-Translation Theorem

Given a preimage graph described by a sentence in x and y, the following two processes yield the same image:
(1) replacing x by $x - h$ and y by $y - k$ in the sentence;
(2) applying the translation $(x, y) \rightarrow (x + h, y + k)$ to the preimage graph.

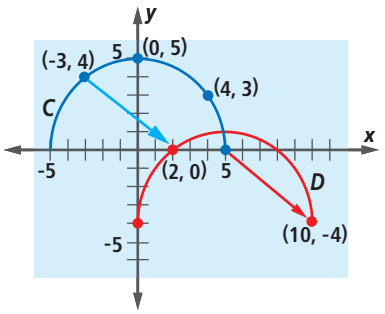
That is, for $T(x, y) = (x + h, y + k)$, an equation of the image of $y = f(x)$ is $y - k = f(x - h)$.

The Graph-Translation Theorem can be applied to write an equation if a graph is given, and to sketch a graph if an equation is given. Translations also occur in situations that might not be described with an equation.

GUIDED

Example 2

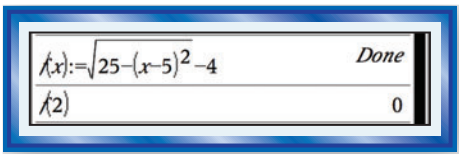
At the right are graphs of the function $y = C(x) = \sqrt{25 - x^2}$ and its image $y = D(x)$ under the translation $(x, y) \rightarrow (x + 5, y - 4)$. Both are semicircles. Find an equation for the image.



Solution Because the translation is 4 units down, replace y with ___?___. Because the translation is 5 units to the right, replace x with ___?___. This gives the equation ___?___. Solve for y. $y =$ ___?___

Check 1 The point (–3, 4) is on C. According to the translation, the image of (–3, 4) is (___?___, ___?___). Show that this image point checks in your equation for D.

Check 2 Use a graphing utility to plot C and D on the same axes.



STOP QY2

▶ **QY2**

What are the domains of C and D?

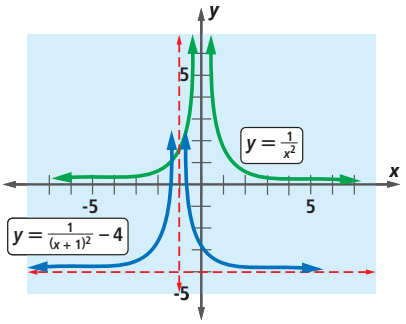
Example 3

Sketch a graph of $y = \frac{1}{(x+1)^2} - 4$.

Solution First rewrite the sentence in the form $y - k = \frac{1}{(x-h)^2}$ to see the replacements in relation to the function $y = \frac{1}{x^2}$.

$$y - -4 = \frac{1}{(x+1)^2} = \frac{1}{(x--1)^2}$$

In the equation $y = \frac{1}{x^2}$, y has been replaced by $y - -4$ and x has been replaced by $x - -1$. Thus, by the Graph-Translation Theorem, the graph of $y - -4 = \frac{1}{(x--1)^2}$ is the image of the graph of $y = \frac{1}{x^2}$ under the translation $T(x, y) = (x - 1, y - 4)$. Therefore, its graph is translated 1 unit to the left and 4 units down from the graph of the parent inverse-square function. In particular, its asymptotes are $x = -1$ and $y = -4$. With this knowledge, its graph can be sketched. The graphs of the parent function and the given equation are drawn at the right.



Questions

COVERING THE IDEAS

In 1–3, find the image of each point under the given translation.

1. $(2, -5)$, move left 3 units and up 6 units
2. $(-14, -7)$, horizontal translation of 2 units and a vertical translation of -3 units
3. (p, q), horizontal translation of a units and vertical translation of b units

In 4 and 5, find the image of the point under T: $(x, y) \rightarrow (x - 3, y + 4)$.

4. $(1, -2)$
5. (r, s)
6. **Multiple Choice** Which rule is for a translation T that has the effect of sliding a graph 3 units down and 7 units to the left?

   A $T(x, y) = (x - 3, y - 7)$ **B** $T(x, y) = (x + 3, y + 7)$

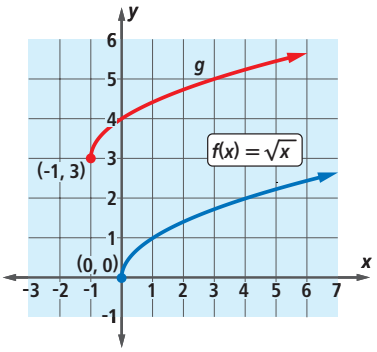
   C $T(x, y) = (x - 7, y + 3)$ **D** $T(x, y) = (x - 7, y - 3)$

7. Suppose $y = \frac{1}{x}$ and $T(x, y) = (x + 3, y + 4)$.

   a. Find the images of $(1, 1)$, $(-1, -1)$, and $(0.5, 2)$ under T.

   b. Verify that the three images under T satisfy $y - 4 = \frac{1}{x - 3}$.

8. The graph of $f(x) = \sqrt{x}$ is shown at the right, together with its image under a translation T. The image of $(0, 0)$ on the graph is the point $(-1, 3)$.
   a. Find a rule for the translation T.
   b. Find the image of $(9, 3)$ under this translation.
   c. Find an equation for the image g.



9. Use the Graph-Translation Theorem to find an equation of the image of $y = |x|$ under $T(x, y) = (x - 4, y + 6)$.

In 10 and 11, rules for two functions are given.
a. State a rule for a translation that maps f onto g.
b. Graph f and g on the same set of axes.

10. $f(x) = \frac{1}{x}$; $g(x) = \frac{1}{x - 2}$

11. $f(x) = x^2$; $g(x) = (x + 2)^2 + 1$

APPLYING THE MATHEMATICS

12. a. Graph f and g, where $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x + 3} - 2$, on the same set of axes.
    b. Find equations for the asymptotes of g. How are they related to the asymptotes of f?
    c. Give the domain and range of f and g.

13. The parabola $y = (x - 3)^2 + 5$ is the image of the parabola $y = (x + 2)^2 + 7$ under a translation T: $(x, y) \rightarrow (x + h, y + k)$. What are the values of h and k?

14. **Multiple Choice** A circle has radius 4 and center $(2, -3)$. Which of the following might be an equation for the circle?
    A $(x + 2)^2 + (y + 3)^2 = 16$
    B $(x - 2)^2 + (y - 3)^2 = 16$
    C $(x - 2)^2 + (y + 3)^2 = 16$
    D $(x + 2)^2 + (y - 3)^2 = 16$

15. The formula $N = 0.82(1.09)^t$ gives the approximate current U.S. national debt (in billions of dollars), t years after 1900.
    a. Compute the estimated national debt for 2007.
    b. Convert the formula to one that maps the actual year y onto the debt.

REVIEW

16. Consider the equation $y = \frac{1}{x - 1}$. **(Lesson 3-1)**
    a. Identify the parent equation.
    b. Graph the equation and its parent on the same axis.

17. Use the mean lengths of largemouth bass of various ages given below. **(Lesson 2-3)**

Age (years)	1	2	3	4	5	6	7	8	9	10
Length (inches)	6.3	9.0	11.6	13.5	15.8	17.4	18.9	19.8	20.3	20.7

Source: Illinois Department of Natural Resources

a. Find an equation of the line of best fit for predicting length from age.
b. Interpret the slope of the line.
c. Use the line to predict the length of a twelve-year-old largemouth bass in Illinois.
d. Suggest a reason for being cautious about your prediction in Part c.
e. Find the correlation coefficient r between the age of the fish and its mean length.
f. Interpret the sign of r.

The world-record largemouth bass, caught in 1932 in Georgia, was 32.5 inches long and 28.5 inches in girth.

18. ***Skill Sequence*** Factor. **(Previous Course)**

a. $k^2 - 9$ b. $9 - 16t^2$ c. $(p + 4)^2 - 25$

EXPLORATION

19. a. Consider the linear equation $f(x) = 3x - 5$. Find an equation for the image of the graph of f under the following transformations.
   i. $T(x, y) = (x + 1, y + 3)$
   ii. $T(x, y) = (x + 2, y + 6)$
   iii. $T(x, y) = (x - 4, y - 12)$
   iv. $T(x, y) = (x + 1, y + 5)$

b. Make a conjecture based on the results of Part a.
c. Prove your conjecture in Part b.
d. Generalize this problem to any line of the form $y = mx + b$.

QY ANSWERS

1. $\{y \mid y \geq 1\}$

2. For C, $-5 \leq x \leq 5$.
For D, $0 \leq x \leq 10$.